Math 118 - Spring 2022 - Common Final Exam, version A Solutions

1. In 2015 the number of people infected by a virus was $P_0 = 325000$. Due to a new vaccine, the number of infected people has decreased by 25% each year since 2015. In other words, only 75% as many people are infected each year as were infected the year before.

   (a) (4 points) Find a formula for the function $P(t)$, the number of infected people t years after 2015 in the form $P(t) = P_0 \cdot b^t$.

   > **Solution:** $P(t) = 325000(0.75)^t$
   >
   > 1 point for the initial value, 3 points for growth factor and setting the formula correctly.

   (b) (4 points) Write the formula for the function $P(t)$ in the form $P(t) = P_0 \cdot e^{kt}$. Round constants in your answer to four decimal places.

   > **Solution:** $P(t) = 325000e^{-0.2877t}$
   >
   > 1 points for the initial value, 3 points for continuous rate and setting the formula correctly.

   (c) (4 points) Find the number of people infected by the virus in 2025. Round your answer to the nearest integer.

   > **Solution:** 18302 people
   >
   > 2 points for realizing that $t = 10$ and plugging into the formula, 2 point for calculations.

2. The tables below contain values from an exponential or a linear function. In each part:

   - Decide if the function is linear or exponential.
   - Find a formula for the function.

   (a) (5 points)

x	0	1	2	3	4
$f(x)$	10.3	11.124	12.01392	12.975	14.013

   > **Solution:** Exponential $f(x) = 10.3(1.08)^x$.
   >
   > 1 point for recognizing an exponential function, 1 point for the initial value, 2 points for growth factor, 1 point for setting the formula correctly

   (b) (5 points)

x	0	1	2	3	4
$i(x)$	30	28	26	24	22

   > **Solution:** Linear $i(x) = 30 - 2x$.
   >
   > 1 point for recognizing a linear function, 1 point for the initial value, 2 points for the slope, 1 point for setting the formula correctly

3. (10 points) An investment decreases by 68% over a 14-year period. At what effective annual percent rate does it decrease? Give your answer as a percentage rounded to two decimal places, like *nn.nn*%.

Solution: The investment decreases by 7.82% each year.

3 points for $P(14) = 0.32 \cdot P_0$
2 points for $P(14) = (1+r)^{14} \cdot P_0$
2 points for setting the equation $0.32 \cdot P_0 = (1+r)^{14} \cdot P_0$
2 points for solving the equation $0.32 = (1+r)^{14}$ for r
1 point for converting decimals to percent: $r = -0.0781642$ means the investment decreases by 7.82% each year.

4. (10 points) Which would earn more money: $30,000$ invested in an account A paying 2% compounded daily (365 times per year), or the same $30,000$ invested in an account B paying 2.1% compounded monthly (12 times per year)? In both cases, suppose the investment lasts for 15 years.

Solution: A: $40495; B: $41096. Option B is better than A

- 4 points for evidence of determining what to compare (balance after 15 years, growth factors)

- 4 points for correct computations

- 2 points for stating a conclusion based on the computations shown

5. (10 points) Find the half-life of a radioactive substance that decays at a continuous rate of 6% per minute. Round your answer to two decimal places.

Solution: 11.55 minutes.

2 points for $Q(T) = 0.5 \cdot Q_0$
3 points for $Q(T) = Q_0 \cdot e^{-0.06 \cdot T}$
2 points for setting the equation $0.5 = e^{-0.06 \cdot T}$
3 points for solving the equation for half-life time $T = \frac{\ln{(0.5)}}{-0.06} = 11.55$.
But some students will be just plugging into the formula from the list provided on the final.

6. A population of animals oscillates sinusodally between a high of 2400 on January 1 $(t = 0)$ and a low of 1800 on April 1 $(t = 3)$.

   (a) (6 points) Find a formula for the population, P, in terms of the time, t, in months.

   Solution: $P(t) = 2100 + 300 \cos\left(\frac{\pi t}{3}\right)$

   - 1 point for finding the midline $= 2100$

   - 1 point for finding the amplitude $= 300$

   - 2 points for finding the period $T = 6$ and parameter $B = \pi/3$

- 1 points for cosine

- 1 point for putting it all together into the formula

(b) (4 points) Graph P as a function of t. Mark two points on your graph where $P = 2000$.
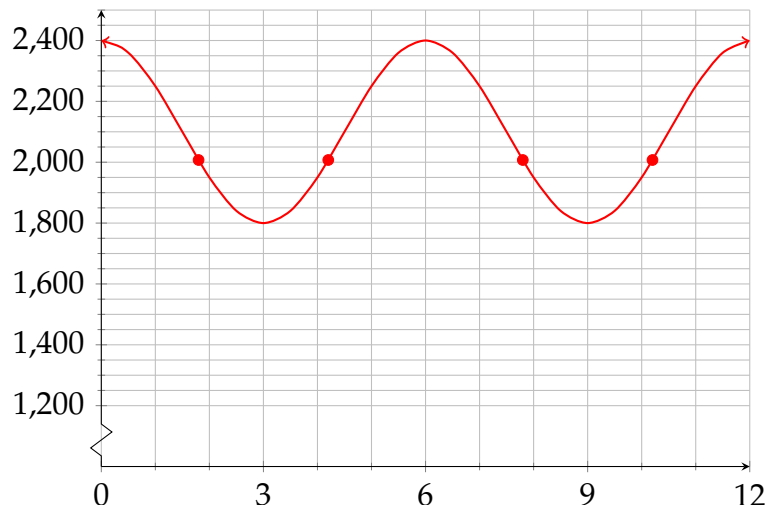
Solution:

- 2 points for correct graph (there aren't really enough points here for a very fine-grained score; 1 points should be for some kind of sinusoidal graph, and 2 points for a substantially correct graph)

- 2 points for plotted points on the graph at $P = 2000$

(c) (4 points) Now use the formula for the population from part (a) to find when the population first reaches 2,000. Give your answer in terms of an inverse trig function, and give your answer rounded to three decimal places with correct units.

Solution: $t = \frac{3 \arccos\left(-\frac{1}{3}\right)}{\pi} \approx 1.83$

1 pt	for setting the equation $2100 + 300 \cos\left(\frac{\pi t}{3}\right) = 2000$
2 pt	for solving in terms of inverse trig function $t = \frac{3 \arccos\left(-\frac{1}{3}\right)}{\pi}$
1 pt	for plugging into a calculator and getting a numerical value $t = 1.825$

7. (10 points) For $\frac{\pi}{2} \leq \theta \leq \pi$ find $\tan(\theta)$ if $\cos(\theta) = -\frac{\sqrt{2}}{3}$. Give the exact answer.

Solution: $\tan(\theta) = -\frac{\sqrt{7}}{\sqrt{2}}$

3 points   for setting correctly the equation $\cos^2(\theta) + \sin^2(\theta) = 1$ and plugging in $\cos(\theta) = -\frac{\sqrt{2}}{3}$

3 points   for solving correctly for $\sin(\theta)$

2 point   for correct sign, so $\sin(\theta) = \frac{\sqrt{7}}{3}$

2 points   for $\tan(\theta) = \frac{\sin(\theta)}{\cos(\theta)}$ (Any equivalent exact answer is OK.)

8. Let $P = f(t) = 14 \cdot 2^{t/12}$ give the size in thousands of an animal population in year t.

  (a) (5 points) Find the inverse function $f^{-1}(P)$.

> **Solution:** $f^{-1}(P) = 12\frac{\ln\left(\frac{P}{14}\right)}{\ln(2)}$.
>
> 2 points   correct algebra other than logarithm
> 2 points   correctly use logarithm
> 1 point   correct expression for f^{-1}

  (b) (5 points) Evaluate $f^{-1}(24)$. Round your answer to two decimal places. In a sentence with correct units, explain what this tells you about the animal population.

> **Solution:** $f^{-1}(24) = 9.33$. The population reaches the size of 24 000 in approximately 9.33 years.
>
> 2 points   correct evaluation of f^{-1}
> 2 points   24 as population, inverse function value as time
> 1 point   correct units

9. The diagram below shows a unit circle (with radius 1). Grid lines are spaced $\frac{1}{10}$ unit apart. The radius corresponding to angle α is shown.
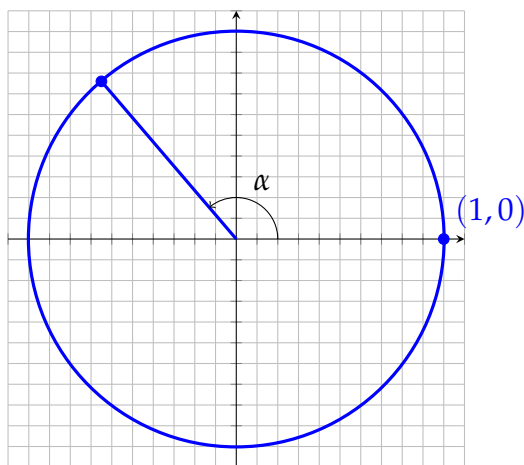
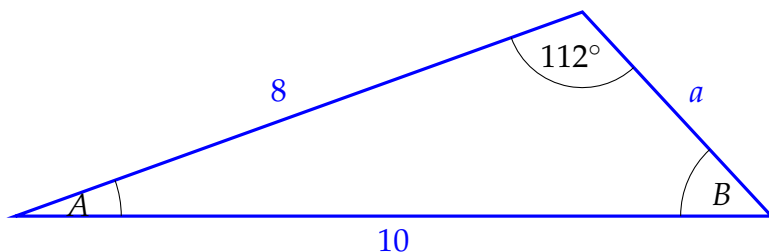 (a) (4 points) Using the grid, estimate $\cos(\alpha)$ and $\sin(\alpha)$ to two decimal places.

- $\cos(\alpha) \approx$ __**-0.65**__
- $\sin(\alpha) \approx$ __**0.76**__

(b) (4 points) Draw another radius on the circle corresponding to an angle in Quadrant III with the same cosine as α.

10. (10 points) Find the missing side and angle measures in the diagram below. Give angles as degrees, and round all answers to one decimal place.

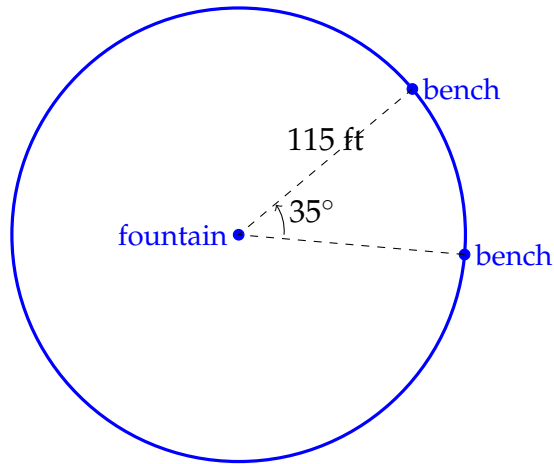11. The diagram below shows the location of two benches at the edge of a circular plaza with a fountain in the middle.

(a) (4 points) The central angle between the benches is $35°$. Give this angle in radians, with an exact answer in terms of π.

> **Solution:** $\frac{7\pi}{36}$.
>
> 3 points   correct conversion, might be given as an approximation
> 1 points   answer is correctly in terms of π

(b) (6 points) The radius of the circular path is 115 feet. Find the length of the short arc between the benches. Give an answer rounded to two decimal places with correct units.
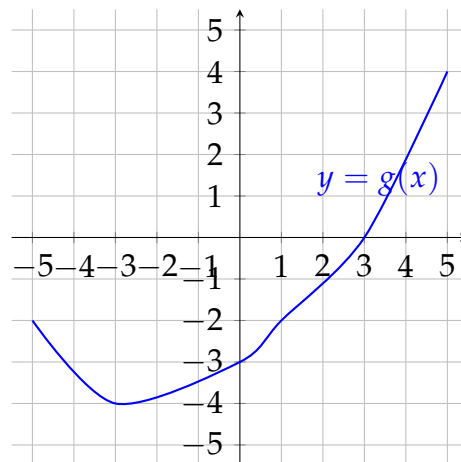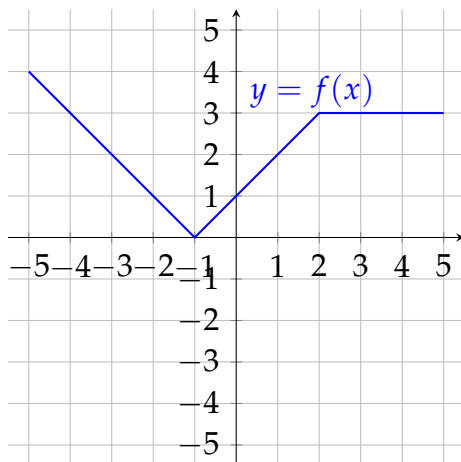
> **Solution:** $\frac{7\pi}{36} \times 115 \approx 70.25$ feet
>
> 2 points   correct proportion or arc length formula
> 2 points   angle measure from previous part correctly substituted
> 2 points   correct computation and units

12. The graphs of $y = f(x)$ and $y = g(x)$ are given below. Use them to evaluate the following compositions.

(a) (4 points) $f\big(g(-3)\big)$

> **Solution:** $f(g(-3)) = f(-4) = 3$.

(b) (4 points) Find all x solving $g(f(x)) = -2$.

> **Solution:** $x = -2$ or $x = 0$.

> **Solution:** For each of these: 2 points for correct evaluation of inside function, 2 points for correct evaluation of outside function.
>
> It is OK to evaluate these without work shown.